

A note on potential energy density in a stratified compressible fluid

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An exact, local, positive definite expression is obtained for the potential energy density in a wide class of stratified compressible fluids. This expression is an extension of that derived for incompressible stratified fluids in the preceding paper by Holliday & McIntyre (1981), and also represents a finite-amplitude analogue of the disturbance potential energy density that is familiar in small-amplitude theory. Its volume integral reduces to Lorenz' (1955) available potential energy under suitable choice of a hydrostatic reference state, provided that the fluid is contained within a fixed volume enclosed by rigid impermeable boundaries.

1. Introduction

In the preceding paper, Holliday & McIntyre (1981) (hereafter designated HM) have derived an exact local expression for the potential energy density in an incompressible stratified fluid; this generalizes the small-amplitude versions for potential energy density in such a fluid that are well known in the literature (for references see HM), and may therefore be useful in discussing motions involving large displacements of fluid particles. The present note extends HM's result to a compressible fluid with an arbitrary equation of state. Under certain restrictions on the equation of state (including the physically reasonable requirement that the fluid shall support sound waves) the potential energy density, which includes gravitational potential energy and internal energy terms, can be demonstrated to be positive definite. The volume integral of the potential energy as defined here is generally greater than the available potential energy defined by Lorenz (1955); however, it does equal the latter when the reference state discussed in § 2 is taken to be Lorenz' reference state, namely that state of hydrostatic equilibrium obtained from the actual instantaneous state by an adiabatic redistribution of mass. The reader is referred to HM for more background discussion: possible applications are mentioned in § 7, below.

2. Derivation of the potential energy density

A complete set of equations describing the inviscid, adiabatic flow of a compressible fluid under gravity are the momentum, continuity and entropy equations,

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + \nabla\Phi = \mathbf{0}, \quad (2.1)$$

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$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0, \quad (2.2)$$

$$\frac{Ds}{Dt} = 0, \quad (2.3)$$

and the equation of state

$$\rho^{-1} = F(s, p). \quad (2.4)$$

Here $\mathbf{u}(\mathbf{x}, t)$ is the fluid velocity, $\rho(\mathbf{x}, t)$ the density, $p(\mathbf{x}, t)$ the pressure, and $s(\mathbf{x}, t)$ the entropy per unit mass; $\Phi(\mathbf{x})$ is the gravitational potential, D/Dt is the material derivative, and F is a suitably well-behaved function. The equations can also be referred to a uniformly rotating frame, if desired, by adding a Coriolis term to (2.1) and redefining Φ to include a centrifugal term in the usual way; the results given below will then be unchanged.

Using (2.1)–(2.4) it is straightforward to derive the energy equation in the usual form

$$\rho \frac{D}{Dt} \left\{ \frac{1}{2} |\mathbf{u}|^2 + \epsilon + \Phi \right\} + \nabla \cdot (p\mathbf{u}) = 0, \quad (2.5)$$

where the internal energy $\epsilon(s, p)$ can be related to the enthalpy $H(s, p)$ by

$$\epsilon(s, p) = H(s, p) - p/\rho = H(s, p) - pF(s, p) \quad (2.6)$$

and the enthalpy itself satisfies

$$H_s = T = G(s, p), \quad H_p = \rho^{-1} = F(s, p), \quad (2.7a, b)$$

where suffixes s and p denote partial derivatives and G represents the thermodynamic dependence of the temperature T on s and p ; H will be assumed twice differentiable in s and p .

We now introduce a reference state \mathcal{R}_0 of hydrostatic equilibrium, in which

$$\mathbf{u} = 0, \quad \{\rho, p, s, T\} = \{\rho_0, p_0, s_0, T_0\},$$

where

$$\frac{1}{\rho_0} \nabla p_0 + \nabla \Phi = 0, \quad (2.8)$$

and

$$\rho_0^{-1} = F(s_0, p_0), \quad T_0 = G(s_0, p_0). \quad (2.9)$$

For the moment we shall not require any relationship between this reference state and the actual state of the fluid; this contrasts with Lorenz' definition of a reference state (\mathcal{R}_L , say) as that obtained from the actual state by an entropy-conserving redistribution of mass.

By taking the curl of (2.8) and using (2.9) we can easily show that all reference state quantities are constant on equipotentials; it turns out to be convenient to express each one as a function of s_0 :

$$\{\rho_0(\mathbf{x}), p_0(\mathbf{x}), T_0(\mathbf{x}), \Phi(\mathbf{x})\} = \{\tilde{\rho}(s_0), \tilde{p}(s_0), \tilde{T}(s_0), \tilde{\Phi}(s_0)\}. \quad (2.10)$$

We assume that in \mathcal{R}_0 the entropy increases 'upwards', i.e. in the direction of increasing Φ , and so

$$\tilde{\Phi}'(s_0) \equiv \frac{d\tilde{\Phi}}{ds_0} > 0. \quad (2.11)$$

The hydrostatic equation (2.8) becomes

$$\tilde{p}'(s_0) = -\tilde{\rho}(s_0)\tilde{\Phi}'(s_0), \tag{2.12}$$

and hence, by (2.11),

$$\tilde{p}'(s_0) < 0. \tag{2.13}$$

With (3.11) below, this implies that \mathcal{R}_0 is statically stable: see §6.2.

We now return to the actual fluid motion expressed by (2.1)–(2.4) and manipulate the energy equation (2.5) into an alternative form. First we note that

$$\nabla \cdot (p_0 \mathbf{u}) = p_0 \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p_0 = -\frac{p_0 D\rho}{\rho Dt} + \frac{Dp_0}{Dt} = \rho \frac{D}{Dt} \left(\frac{p_0}{\rho} \right); \tag{2.14}$$

the second equality uses the continuity equation (2.2) and the fact that $\partial p_0 / \partial t = 0$. We next invoke the thermodynamic identities (2.7) to obtain

$$\frac{D}{Dt} H(s_0, p_0) = G(s_0, p_0) \frac{Ds_0}{Dt} + F(s_0, p_0) \frac{Dp_0}{Dt},$$

and hence, using (2.9), (2.10) and also $\partial p_0 / \partial t = 0$ again,

$$\begin{aligned} \frac{D}{Dt} H(s_0, p_0) &= \tilde{T}(s_0) \frac{Ds_0}{Dt} + \rho_0^{-1} \mathbf{u} \cdot \nabla p_0 \\ &= \frac{D}{Dt} \int_{s_0}^s \tilde{T}(s') ds' - \mathbf{u} \cdot \nabla \Phi, \end{aligned} \tag{2.15}$$

by (2.8). Using (2.3), we can insert s as the lower limit in the integral in (2.15) provided that $\tilde{T}(s')$ is defined for all s' lying between s_0 and s . We ensure that the latter condition holds by requiring that every value of the entropy in the actual state shall also be present in the reference state; then

$$\frac{D}{Dt} \Phi(\mathbf{x}) \equiv \mathbf{u} \cdot \nabla \Phi = -\frac{D}{Dt} \left\{ \int_{s_0}^s \tilde{T}(s') ds' + H(s_0, p_0) \right\}. \tag{2.16}$$

Finally, we substitute (2.6), (2.14) and (2.16) into the energy equation (2.5) and obtain

$$\rho \frac{D}{Dt} \left\{ \frac{1}{2} |\mathbf{u}|^2 + \Pi \right\} + \nabla \cdot \{ (p - p_0) \mathbf{u} \} = 0, \tag{2.17}$$

where

$$\Pi\{s, p; s_0, p_0\} \equiv H(s, p) - H(s_0, p_0) - \frac{(p - p_0)}{\rho} - \int_{s_0}^s \tilde{T}(s') ds' \tag{2.18}$$

is a potential energy per unit mass.† It is important to notice that the derivation of (2.17) from (2.5) has not employed the momentum equation (2.1). We also observe that the *excess* pressure $p - p_0$, rather than the *total* pressure p , appears in the divergence term in (2.17), just as it does in the ‘disturbance energy equation’ for small-amplitude acoustic-gravity waves (e.g. Lighthill 1978, §§ 1.3, 4.2). This helps explain why, in the limit of small-amplitude disturbances to a resting fluid, Π reduces to the potential energy density found in the linear theory; see § 6.2, below.

† Note that HM work in terms of a potential energy per unit volume, E_p , while we find it more convenient in the compressible case to use a potential energy per unit mass.

3. The positive definite character of Π

From the definition (2.18) it is clear that Π vanishes when $s = s_0$ and $p = p_0$; we now show that Π is positive if $(s, p) \neq (s_0, p_0)$. To do this we split Π into two parts:

$$\Pi = \Pi_1 + \Pi_2, \quad (3.1)$$

where

$$\Pi_1 = H(s, p) - H(s, p_0) - (p - p_0)/\rho, \quad (3.2)$$

and

$$\Pi_2 = H(s, p_0) - H(s_0, p_0) - \int_{s_0}^s \hat{T}(s') ds'. \quad (3.3)$$

We first consider Π_1 : using (2.7b) and some integrations by parts, it can easily be shown that

$$\Pi_1 = - \int_{p_0}^p (p' - p_0) H_{pp}(s, p') dp'. \quad (3.4)$$

Equation (2.7b) also gives

$$H_{pp}(s, p) = F_p = -(\rho c)^{-2}, \quad (3.5)$$

where

$$c^2(s, p) \equiv \left[\frac{\partial p(\rho, s)}{\partial \rho} \right]_{\rho^{-1} = F(s, p)};$$

we shall assume that

$$c^2 > 0 \quad (3.6)$$

for all s and p ; then c is the speed of sound. By (3.5) and (3.6) $H_{pp} < 0$ for all s and p ; in particular

$$-H_{pp}(s, p') > 0, \quad (3.7)$$

where p' lies between p_0 and p . Consideration of the two alternatives $p \geq p_0$ and use of (3.7) then show that the integral in (3.4) is positive definite; that is

$$\Pi_1 > 0 \quad \text{if } p \neq p_0. \quad (3.8)$$

We next turn attention to Π_2 ; (3.3) yields

$$\begin{aligned} \Pi_2 &= \int_{s_0}^s \{H_s(s', p_0) - \hat{T}(s')\} ds' \\ &= \int_{s_0}^s \{G(s', \tilde{p}(s_0)) - G(s', \tilde{p}(s'))\} ds', \end{aligned} \quad (3.9)$$

by (2.7a), (2.9) and (2.10). Now (2.13) implies that \tilde{p} is a monotonically decreasing function of its argument, so that

$$\tilde{p}(s_0) \geq \tilde{p}(s') \quad \text{if } s' \geq s_0. \quad (3.10)$$

If we make the further assumption that

$$G_p(s, p) \equiv H_{sp}(s, p) > 0, \quad (3.11)$$

for all s and p , we find that

$$G(s', \tilde{p}(s_0)) - G(s', \tilde{p}(s')) \geq 0 \quad \text{if } s' \geq s_0,$$

and hence the last integral in (3.9) is non-negative. Thus

$$\Pi_2 > 0 \quad \text{if } s \neq s_0. \tag{3.12}$$

From (3.8), (3.12) and (3.1) it then follows that

$$\Pi > 0 \quad \text{if } (s, p) \neq (s_0, p_0); \tag{3.13}$$

this completes the proof that Π is positive definite.

We note that the thermodynamic inequality (3.11) can be shown to hold if $c^2 > 0$ and $\partial p(\rho, s)/\partial s > 0$; in particular it is satisfied by a perfect gas, as may be verified from (6.2) below.

It should also be observed that the expressions (3.4) for Π_1 and (3.9) for Π_2 depend only on the reference state, the thermodynamic function H and its derivatives, and the local quantities s and p ; they are natural generalizations of (2.15) of HM. Moreover, in the incompressible limit, where $c^2 \rightarrow \infty$ and ρ becomes a function of s alone, it can be verified that $\Pi_1 \rightarrow 0$ and $\rho\Pi_2 \rightarrow E_p$ as given by (2.15) of HM.

4. Physical interpretation of Π

To obtain a physical interpretation of Π analogous to that in HM §3, it is first necessary to rearrange (2.18) once more. Note that in the reference state

$$H(s_0, p_0) = H(s_0, \tilde{p}(s_0)) \equiv \tilde{H}(s_0), \tag{4.1}$$

say. Then

$$\tilde{H}'(s_0) = \tilde{T}'(s_0) + \tilde{p}'(s_0)/\tilde{\rho}(s_0) = \tilde{T}'(s_0) - \tilde{\Phi}'(s_0)$$

by (2.7), (2.9), (2.10) and (2.12), and so

$$\begin{aligned} \int_{s_0}^s \tilde{T}'(s') ds' &= [\tilde{H}(s') + \tilde{\Phi}(s')]_{s_0}^s \\ &= H(s, \tilde{p}(s)) - H(s_0, p_0) + \tilde{\Phi}(s) - \tilde{\Phi}(s_0), \end{aligned} \tag{4.2}$$

using (4.1). From (2.6) we have

$$\begin{aligned} \epsilon(s, \tilde{p}(s)) &= H(s, \tilde{p}(s)) - \tilde{p}(s) F(s, \tilde{p}(s)) \\ &= H(s, \tilde{p}(s)) - \tilde{p}(s)/\tilde{\rho}(s), \end{aligned} \tag{4.3}$$

using also (2.9) and (2.10). Substitution of (4.2), (4.3) and (2.6) into (2.18) then yields

$$\Pi = \{\tilde{\Phi}(s_0) - \tilde{\Phi}(s)\} + \{\epsilon(s, p) - \epsilon(s, \tilde{p}(s))\} + \{(p_0/\rho) - (\tilde{p}(s)/\tilde{\rho}(s))\}. \tag{4.4}$$

Consider a fluid particle P of unit mass, currently at (\mathbf{x}, t) ; its entropy is $s = s(\mathbf{x}, t)$, its pressure is $p = p(\mathbf{x}, t)$ and its internal energy is $\epsilon = \epsilon(s, p)$. Provided that s lies within the range of values of entropy that exist in the reference state (this was assumed above, in (2.16)), we can also find a particle of unit mass at position \mathbf{x}_0 in the reference state which has the same entropy as P , i.e.

$$s_0(\mathbf{x}_0) = s(\mathbf{x}, t). \tag{4.5}$$

The simplest physical situation in which this can be done is when an entropy- and mass-conserving motion taking the reference state to the actual state can be found; then \mathbf{x}_0 is the initial position of particle P . For the purpose of this discussion we therefore suppose that an external agency does indeed redistribute particles

adiabatically in this way, and we compute the work W that is done on P while it is being transferred from the position \mathbf{x}_0 in the reference state to the position \mathbf{x} in the actual state, along some path C .

The presence of the excess pressure $p - p_0$ in the energy equation (2.17) suggests that a physical interpretation of Π must take account of compressional work done on P by the *excess* pressure $p - p_0$ alone: this is the same philosophy as adopted by Lighthill (1978, §1.3) for small-amplitude disturbances. We must likewise consider the work done against the basic pressure gradient force $-\nabla p_0$. Thus W includes a contribution

$$W_1 = \int_C (p - p_0) (-d\rho^{-1}), \quad (4.6)$$

that represents the work done by the excess pressure in compressing P ; there is also a second contribution

$$W_2 = \int_C \{\nabla\Phi + \rho^{-1}\nabla p_0\} \cdot d\mathbf{x} \quad (4.7)$$

representing the energy gained by P as it is moved through the unbalanced force field $-\rho\nabla\Phi - \nabla p_0$.

Now for adiabatic processes $ds = 0$, and so $-pd\rho^{-1} = d\epsilon$ by (2.6) and (2.7); thus

$$\begin{aligned} W = W_1 + W_2 &= \int_C \{d\epsilon + d(p_0\rho^{-1}) + d\Phi\} \\ &= [\epsilon + p_0\rho^{-1} + \Phi]_{\mathbf{x}_0}^{\mathbf{x}}. \end{aligned} \quad (4.8)$$

(Note that the parts of W_1 and W_2 involving p_0 in (4.6) and (4.7) each depend on the path C ; however, these add to give a path-independent term in W .) By (2.10), $\Phi(\mathbf{x}) = \tilde{\Phi}(s_0(\mathbf{x}))$ and, using (4.5) as well, $\Phi(\mathbf{x}_0) = \tilde{\Phi}(s)$. Moreover, since the entropy at \mathbf{x}_0 in the reference state is $s_0(\mathbf{x}_0) = s$, the pressure, density and internal energy at \mathbf{x}_0 in the reference state must be $\tilde{p}(s)$, $\tilde{\rho}(s)$ and $\epsilon(s, \tilde{p}(s))$, respectively. It can then be seen from (4.4) and (4.8) that $W = \Pi$, showing that Π equals the work done on P in bringing it from its initial position in the reference state to its current location.

5. Connection with Lorenz' available potential energy

We now consider the fluid to be contained within a fixed volume V bounded by a rigid impermeable surface S . We define

$$\bar{\Pi} \equiv \int_V \rho \Pi dV, \quad \geq 0 \quad (5.1)$$

and note that

$$\frac{d\bar{\Pi}}{dt} = \int_V \rho \frac{D\Pi}{Dt} dV. \quad (5.2)$$

If the derivation of (2.17) from (2.5) is now retraced, it is easy to see that, for a motion that conserves mass (equation (2.2)) and entropy (equation (2.3)),

$$\rho \frac{D\Pi}{Dt} = \rho \frac{D}{Dt} (\epsilon + \Phi) + \nabla \cdot (p_0 \mathbf{u}). \quad (5.3)$$

In particular, as remarked in §2, the derivation of (5.3) does not use the momentum equation (2.1) and would hold even if an arbitrary body force were included in (2.1).

Since S is rigid and impermeable, (5.2), (5.3) and the divergence theorem give

$$\frac{d\bar{\Pi}}{dt} = \frac{d}{dt} \int_V \rho(\epsilon + \Phi) dV = \frac{dP_T}{dt}, \tag{5.4}$$

where

$$P_T \equiv \int_V \rho(\epsilon + \Phi) dV, \tag{5.5}$$

is the total potential energy (Lorenz 1955). Time integration of (5.4) from some initial state (denoted by the suffix i) gives

$$\bar{\Pi} = (P_T - P_{T1}) + \bar{\Pi}_1, \tag{5.6}$$

where

$$\bar{\Pi}_1 \geq 0. \tag{5.7}$$

We drop the assumption used in the previous section, that the actual state be attainable by an adiabatic mass redistribution of the reference state \mathcal{R}_0 . Instead we consider a (possibly hypothetical) mass- and entropy-conserving motion $\mathbf{u}(\mathbf{x}, t)$ in which the initial state is Lorenz' reference state \mathcal{R}_L ; by definition of \mathcal{R}_L we know that such a velocity field can always be found, although it need not satisfy the momentum equation (2.1). Equations (5.3), (5.4) and (5.6) still hold for such a motion, however, while by definition

$$P_T - P_{T1} = A, \tag{5.8}$$

A being Lorenz' available potential energy. Hence by (5.6) and (5.7)

$$\bar{\Pi} = A + \bar{\Pi}_1 \geq A. \tag{5.9}$$

Thus the volume integral of our potential energy generally exceeds the available potential energy. However, if we choose our reference state \mathcal{R}_0 to be the same as Lorenz' (\mathcal{R}_L), $\bar{\Pi}_1 = 0$ and then $\bar{\Pi}$ does equal A .

6. Two special cases

6.1. A perfect gas

We briefly present results for this case, which is commonly used in meteorology and other fields. For such a gas

$$\rho^{-1} = F(s, p) = (\kappa c_p) e^{s/c_p} p^{-(1-\kappa)}, \quad T = G(s, p) = e^{s/c_p} p^\kappa \tag{6.1 a, b}$$

and

$$H = c_p e^{s/c_p} p^\kappa \tag{6.2}$$

where c_p is the specific heat at constant pressure (a positive constant) and κ is the gas constant divided by c_p , and lies between 0 and 1; we have normalized the pressure by a standard value for convenience. Recalling (3.2), we readily derive

$$\Pi_1 = c_p e^{s/c_p} p_0^\kappa f(p/p_0) \tag{6.3}$$

where

$$f(\xi) \equiv (1 - \kappa) \xi^\kappa + \kappa \xi^{-(1-\kappa)} - 1. \tag{6.4}$$

For $\xi \geq 0$ it is easily shown that

$$f(1) = 0; \quad f(\xi) > 0 \quad \text{for} \quad \xi \neq 1 \tag{6.5}$$

(since $0 < \kappa < 1$), thus immediately verifying that $\Pi_1 > 0$ for $p \neq p_0$, as in (3.8). Using (3.3), we also find

$$\Pi_2 = \int_{s_0}^s \{p_0^\kappa e^{s'/c_p} - \tilde{T}(s')\} ds',$$

where $\tilde{T}(s') = e^{s'/c_p} [\tilde{p}(s')]^\kappa$, and so

$$\Pi_2 = \int_{s_0}^s e^{s'/c_p} \{[\tilde{p}(s_0)]^\kappa - [\tilde{p}(s')]^\kappa\} ds'. \quad (6.6)$$

Since \tilde{p} is a monotonically decreasing function (by (2.13)), (6.6) implies that $\Pi_2 > 0$ for $s \neq s_0$, as in (3.12). For the perfect gas, Π is the sum of the expressions (6.3) and (6.6).

Specializing further to the case of a perfect gas with an isothermal reference state ($T_0 = \text{constant}$), we obtain

$$\tilde{T}(s_0) = T_0 = \text{constant},$$

and

$$\tilde{p}(s_0) = p_0 = T_0^{1/\kappa} \exp(-s_0/\kappa c_p)$$

from (6.1*b*). Then (6.3) becomes

$$\Pi_1 = c_p T_0 \exp\{(s - s_0)/c_p\} f(p/p_0)$$

and (6.6) can be integrated to give

$$\Pi_2 = c_p T_0 h((s - s_0)/c_p),$$

where

$$h(\eta) \equiv e^\eta - 1 - \eta.$$

It is easy to check directly that $h(\eta)$ is non-negative for all η , which is again consistent with previous results.

6.2. Small disturbances

If we assume that the actual fluid flow represents a small departure from the reference state \mathcal{R}_0 , it is straightforward to verify that Π reduces to the standard expression, valid to second order in small quantities, for the potential energy density of infinitesimal acoustic-gravity disturbances about a state of rest (cf. Lighthill 1978, equation (4.37)). In the first place, with sufficient accuracy we may replace s and p' by s_0 and p_0 , respectively, in the term $H_{pp}(s, p')$ in (3.4) to obtain

$$\Pi_1 \doteq \frac{1}{2}(p - p_0)^2 / (\rho_0^2 c_0^2), \quad (6.7)$$

using (3.5), where $c_0 \equiv c(s_0, p_0)$. Equation (6.7) is the small-amplitude expression for the acoustic, or compressional, potential energy per unit mass. Likewise, expansion of the integrand in (3.9) gives

$$\Pi_2 \doteq -\frac{1}{2}(s - s_0)^2 \tilde{p}'(s_0) G_p(s_0, \tilde{p}(s_0)), \quad (6.8)$$

to leading order in $s - s_0$. If gravity is assumed constant and unidirectional, we can write $\Phi = gz$, z being a vertical co-ordinate and g a constant, and also $s_0 = s_0(z)$. A vertical particle displacement ζ can be defined approximately by

$$s - s_0 \doteq -\zeta ds_0/dz, \quad (6.9)$$

and a squared buoyancy frequency by

$$N^2(z) = -\frac{g}{\rho_0} \frac{d\rho_0}{dz} - \frac{g^2}{c_0^2}, \quad (6.10)$$

as usual. Straightforward manipulations, including use of the relation

$$\frac{d}{ds_0} \{H_p(s_0, \tilde{p}(s_0))\} = H_{ps}(s_0, \tilde{p}(s_0)) + H_{pp}(s_0, \tilde{p}(s_0)) \tilde{p}'(s_0)$$

and (2.7), (2.12), (3.5), and (6.8)–(6.10), together with a change of independent variable from s_0 to z , then yield

$$\Pi_2 \doteq \frac{1}{2} N^2 \zeta^2, \tag{6.11}$$

the standard small-amplitude approximation to the gravitational potential energy per unit mass. Note that (2.13) and (3.11) imply that $N^2 > 0$, the usual condition for static stability of \mathcal{R}_0 to small disturbances.

7. Discussion

The quantity Π introduced in this note is only one of a number of possible expressions representing the potential-plus-internal energy density that can be derived for an inviscid, adiabatic, compressible fluid. However, we believe that it has several conceptual and perhaps practical advantages over other, more common, forms. First, it is defined locally, and thus can be used in interpretation of local energy transfers, in contrast to Lorenz' available potential energy, which is defined only globally. Secondly, it is positive definite everywhere, and is therefore a natural generalization to finite amplitude of the disturbance potential energy density used in the theory of infinitesimal disturbances: we have shown that it reduces to the latter in the limit of small amplitude. The property of non-negativeness could represent a useful conceptual advantage of Π over a naive definition of a local disturbance potential energy density as

$$\rho^{-1} \{ \rho [\epsilon(s, p) + \Phi(\mathbf{x})] - \rho_0 [\epsilon(s_0, p_0) + \Phi(\mathbf{x})] \},$$

which can take negative values.

In practical cases $\bar{\Pi}$ could be a useful tool in the interpretation of the global energetics of numerical studies of large-amplitude disturbances in compressible fluids like the atmosphere (e.g Simmons & Hoskins 1978). The 'exact' expression for the available potential energy A , presented by Lorenz (1955, equation (4); 1967, equation (109)) and valid for flows with hydrostatic balance in the vertical, is not generally suitable for computational purposes. Various approximate versions are usually used in numerical models and observational studies of the atmosphere, or alternative approaches are adopted (cf. Pearce 1978). The difficulty of evaluating the exact form of A is related to the difficulty of constructing Lorenz' reference state \mathcal{R}_L , given the actual distributions of mass and entropy in the fluid. Rather than using approximations to A it may be preferable in numerical simulations to choose some simple reference state \mathcal{R}_0 so that Π and $\bar{\Pi}$ are readily calculable exactly, even though $\bar{\Pi}$ is not minimal in the sense that A is. Careful choice of \mathcal{R}_0 might still ensure that $\bar{\Pi}$ is much less than the total potential energy P_T , most of which, as pointed out by Lorenz, may be irrelevant to dynamical discussions. [It should be noted that for a real fluid, including diabatic effects, there is no fundamental reason for preferring Lorenz' reference state to any other: cf. Lorenz (1967), pp. 102–103.] In observational studies of the atmosphere, however, such possible advantages of $\bar{\Pi}$ over the approximate value of A could be masked by inaccuracies in the data (N.-C. Lau, personal communication, 1980).

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